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Scattering of relativistic particles with Aharonov–Bohm–Coulomb interaction in two dimensions

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Abstract. The Aharonov–Bohm–Coulomb potentials in two dimensions may describe the interaction between two particles carrying electric charge and magnetic flux, say, Chern–Simons solitons, or so-called anyons. The scattering problem for such two-body systems is extended to the relativistic case, and the scattering amplitude is obtained as a partial wave series. The electric charge and magnetic flux is $(-q, -\phi/Z)$ for one particle and (Zq, ϕ) for the other. When $(Zq^2/\hbar c)^2 \ll 1$, and $q\phi/2\pi\hbar c$ takes on integer or half-integer values, the partial wave series is summed up approximately to give a closed form. The results exhibit some nonperturbative features and cannot be obtained from perturbative quantum electrodynamics at the tree level.

In a recent letter, we have studied the scattering of relativistic electrons (or positrons) by the Coulomb field of a nucleus in two dimensions [1]. The Dirac equation was solved in polar coordinates, and the scattering amplitude was obtained as a partial wave series. For light nuclei the series can be summed up approximately to give a closed result. The result, though being approximate, exhibits some nonperturbative features and cannot be obtained from the lowest-order contribution of perturbative quantum electrodynamics (QED). This feature is not manifest in the corresponding result in three dimensions. The purpose of this paper is to extend our previous work to the case with both Aharonov–Bohm (AB) and Coulomb potentials. In the nonrelativistic case, this can be described by the stationary Schrödinger equation,

$$-\frac{\hbar^2}{2\mu}\left(\nabla + \mathrm{i}\frac{q\phi}{2\pi\hbar c}\nabla\theta\right)^2\psi - \frac{Zq^2}{r}\psi = E\psi \tag{1}$$

where (r, θ) are polar coordinates on the xy plane, and has been studied in the literature [2,3]. Note that this is a two-dimensional model and is different from the three-dimensional Aharonov–Bohm–Coulomb (ABC) system studied in the literature [4–9]. Of course the AB potential is the same in either two or three dimensions. The difference lies in the Coulomb field. Previously the ABC system (in two or three dimensions) was regarded as describing a charged particle (with charge $-q$) moving in both an AB (with flux ϕ) and a Coulomb field generated by external sources. Thus if the Coulomb field is generated by a nucleus (with charge Zq), one should have a magnetic flux string fixed on the nucleus to generate the AB

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potential. The situation seems difficult to realize in practice. In a recent paper [3], we have shown that the above equation can describe the relative motion of two particles, one carrying electric charge and magnetic flux $(-q,\ -\phi/Z)$ and the other $(Zq,\ \phi)$, where $Z \neq 0$ is a real number. Then μ is the reduced mass of the system. Such particles appear in (2 + 1)-dimensional Chern–Simons field theories as charged vortex soliton solutions [10–17]. They are also called anyons because their angular momentum may take on values other than integers or half-integers. Applications of such objects can be found in the study of the fractional quantum Hall effect [18–21], superconductivity [21, 22] and repulsive Bose gases [23]. Of course the solitons have finite sizes, and the real interaction between them may be rather complicated. Thus the above description by the two-dimensional ABC system is merely a rough approximation. In any case, the model may be of interest in itself since exact analysis is possible. A relativistic case described by the Klein–Gordon equation has been studied in the literature [24]. Here we will deal with a spin-$\frac{1}{2}$ particle described by the Dirac equation.

Let us begin with the stationary Dirac equation translated from equation (1):

$$\left[c\boldsymbol{\alpha}\cdot\left(\boldsymbol{p}+\frac{q\phi}{2\pi c}\nabla\theta\right)+\gamma^0\mu c^2-\frac{\kappa}{r}\right]\psi = E\psi \tag{2}$$

where $\kappa = Zq^2$, $\boldsymbol{p}$ is the momentum operator, $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$ and $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$ $(\mu = 0, 1, 2)$ are Dirac matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ where $g^{\mu\nu} = \mathrm{diag}(1, -1, -1)$. In two dimensions the Dirac matrices can be realized by 2×2 matrices:

$$\gamma^0 = \sigma^3 \qquad \gamma^1 = \mathrm{i}\sigma^1 \qquad \gamma^2 = \mathrm{i}\sigma^2 \tag{3}$$

where the σ are Pauli matrices. Thus ψ is a two-component spinorial wavefunction. As in the case with a pure Coulomb field, it is not difficult to show that the conserved total angular momentum operator is

$$J = xp_y - yp_x + \frac{\mathrm{i}\hbar}{2}\gamma^1\gamma^2. \tag{4}$$

Therefore the particle has spin $\frac{1}{2}$.

The Dirac equation (2) can be solved in the polar coordinates by separation of variables. Bound-state solutions (for $Z > 0$ or $\kappa > 0$) are relatively easy to obtain, and some results will be given at the end of the paper. Scattering solutions exist when $E > \mu c^2$ or $E < -\mu c^2$. The latter correspond to antiparticles after second quantization. At the level of single-particle theory, their scattering can be treated formally in a way similar to the former. Thus we only consider scattering solutions with $E > \mu c^2$ in the following. We use the representation (3). Let

$$\frac{q\phi}{2\pi\hbar c} = m_0 + \nu \tag{5}$$

where m_0 is an integer and $-\frac{1}{2} < \nu \leqslant \frac{1}{2}$, and

$$\psi_j(r,\theta) = \begin{pmatrix} f(r)\exp[\mathrm{i}(j-m_0-\frac{1}{2})\theta]/\sqrt{r} \\ g(r)\exp[\mathrm{i}(j-m_0+\frac{1}{2})\theta]/\sqrt{r} \end{pmatrix} \qquad j = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \ldots. \tag{6}$$

It is easy to show that $J\psi_j = (j - m_0)\hbar\psi_j$. Thus j is a good quantum number. Substituting this expression into equation (2) we obtain two coupled ordinary differential equations for the radial wavefunctions:

$$\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{j+\nu}{r}f + k_1 g + \frac{\gamma}{r}g = 0 \tag{7a}$$

$$\frac{\mathrm{d}g}{\mathrm{d}r} + \frac{j+\nu}{r}g - k_2 f - \frac{\gamma}{r}f = 0 \tag{7b}$$

where

$$k_1 = \frac{E + \mu c^2}{\hbar c} \qquad k_2 = \frac{E - \mu c^2}{\hbar c} \qquad \gamma = \frac{\kappa}{\hbar c} = \frac{Zq^2}{\hbar c}. \tag{8}$$

Then we introduce the new variable

$$\rho = kr \qquad k = \sqrt{k_1 k_2} = \frac{\sqrt{E^2 - \mu^2 c^4}}{\hbar c} \tag{9}$$

and two new functions $u(\rho)$, $v(\rho)$ through the definition

$$f(r) = \tfrac{1}{2}\sqrt{k_1}\mathrm{e}^{\mathrm{i}\rho}[u(\rho) + v(\rho)] \qquad g(r) = -\frac{\mathrm{i}}{2}\sqrt{k_2}\mathrm{e}^{\mathrm{i}\rho}[u(\rho) - v(\rho)] \tag{10}$$

to recast equations (7) into the form

$$\frac{\mathrm{d}u}{\mathrm{d}\rho} - \frac{\mathrm{i}\beta}{\rho}u - \frac{\mathrm{i}\beta' + j + \nu}{\rho}v = 0 \tag{11a}$$

$$\frac{\mathrm{d}v}{\mathrm{d}\rho} + 2\mathrm{i}v + \frac{\mathrm{i}\beta}{\rho}v + \frac{\mathrm{i}\beta' - j - \nu}{\rho}u = 0 \tag{11b}$$

where

$$\beta = \frac{\gamma}{2}\left(\sqrt{\frac{k_1}{k_2}} + \sqrt{\frac{k_2}{k_1}}\right) = \frac{\kappa}{\hbar v_\mathrm{c}} \tag{12a}$$

$$\beta' = \frac{\gamma}{2}\left(\sqrt{\frac{k_1}{k_2}} - \sqrt{\frac{k_2}{k_1}}\right) = \beta\sqrt{1 - \frac{v_\mathrm{c}^2}{c^2}} \tag{12b}$$

where v_c is the classical velocity of the incident particle. This is more convenient. Indeed, one can eliminate v immediately to obtain an equation for u alone:

$$\rho\frac{\mathrm{d}^2u}{\mathrm{d}\rho^2} + (1 + 2\mathrm{i}\rho)\frac{\mathrm{d}u}{\mathrm{d}\rho} + \left(2\beta - \frac{(j+\nu)^2 - \gamma^2}{\rho}\right)u = 0 \tag{13}$$

where we have used $\beta^2 - \beta'^2 = \gamma^2$. In this paper we assume for convenience that $|\gamma| < \frac{1}{2}$ (for electron–nucleus interaction, this means that $Z \leqslant 68$, which is in general satisfied in practice). This is not quite enough. If $|\nu|$ is not close to $\frac{1}{2}$, we further assume that $|\gamma| < |\frac{1}{2} \pm \nu|$. (This cannot hold if $|\nu|$ is very close to $\frac{1}{2}$, which causes some difficulty and will be discussed separately in the following. A larger γ causes more difficulty.) Then for any j, the solution of equation (13) is well behaved at the origin. Let

$$u(\rho) = \rho^s w(\rho) \qquad s = \sqrt{(j+\nu)^2 - \gamma^2}. \tag{14}$$

Then we have for w the equation

$$\rho\frac{\mathrm{d}^2w}{\mathrm{d}\rho^2} + (2s + 1 + 2\mathrm{i}\rho)\frac{\mathrm{d}w}{\mathrm{d}\rho} + 2(\beta + \mathrm{i}s)w = 0. \tag{15}$$

This is familiar. The solution that is well behaved at the origin is $w(\rho) = \Phi(s - \mathrm{i}\beta, 2s + 1, -2\mathrm{i}\rho)$, where $\Phi(a, b, z)$ is the confluent hypergeometric function [25]. So we have

$$u_j(\rho) = a_j\rho^s\Phi(s - \mathrm{i}\beta, 2s + 1, -2\mathrm{i}\rho) \tag{16a}$$

where a_j is a constant, and the subscript j of u_j that is omitted above has been recovered. Substituting this solution into equation (11a) we have

$$v_j(\rho) = a_j\frac{s - \mathrm{i}\beta}{j + \nu + \mathrm{i}\beta'}\rho^s\Phi(s + 1 - \mathrm{i}\beta, 2s + 1, -2\mathrm{i}\rho) \tag{16b}$$

where we have used the formula

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z}+a\right)\Phi(a,b,z)=a\Phi(a+1,b,z)$$

which can be obtained from other relations given in mathematical handbooks [25]. It should be remarked that ψ_j may be slightly singular at the origin when $j=\pm\frac{1}{2}$. However, the integral of $\psi_j^\dagger\psi_j$ over any finite volume converges and the solution is acceptable. This situation is similar to the case of a pure Coulomb field [1]. We take

$$a_j=A2^s(j+\nu+\mathrm{i}\beta')\frac{\Gamma(s-\mathrm{i}\beta)}{\Gamma(2s+1)}\exp\left(\frac{\beta\pi}{2}+\mathrm{i}m\pi-\mathrm{i}\frac{s\pi}{2}+\mathrm{i}\frac{\pi}{4}\right) \tag{17a}$$

where $m=j-\frac{1}{2}$ and

$$A=\mathrm{i}\sqrt{\frac{2}{\pi k}}\frac{1}{\sqrt{k_1+k_2}} \tag{17b}$$

then when $r\to\infty$ we have for the radial wavefunctions the asymptotic forms

$$\begin{aligned}f_j(r)\to A\sqrt{k_1}\bigg[&\mathrm{i}^m\cos\left(kr+\beta\ln 2kr-\frac{m\pi}{2}-\frac{\pi}{4}\right)\\&+\frac{1}{2}(S_j-1)\exp\left(\mathrm{i}kr+\mathrm{i}\beta\ln 2kr-\mathrm{i}\frac{\pi}{4}\right)\bigg]\end{aligned} \tag{18a}$$

$$\begin{aligned}g_j(r)\to A\sqrt{k_2}\bigg[&\mathrm{i}^m\sin\left(kr+\beta\ln 2kr-\frac{m\pi}{2}-\frac{\pi}{4}\right)\\&+\frac{1}{2\mathrm{i}}(S_j-1)\exp\left(\mathrm{i}kr+\mathrm{i}\beta\ln 2kr-\mathrm{i}\frac{\pi}{4}\right)\bigg]\end{aligned} \tag{18b}$$

to the lowest order, where

$$S_j=\exp(2\mathrm{i}\eta_j)=(j+\nu+\mathrm{i}\beta')\frac{\Gamma(s-\mathrm{i}\beta)}{\Gamma(s+1+\mathrm{i}\beta)}\exp(\mathrm{i}j\pi-\mathrm{i}s\pi) \tag{19}$$

and the η_j are phase shifts. The asymptotic form for $\psi=\sum_j\psi_j$, where the summation is taken over all j, turns out to be

$$\psi\to\psi_{\mathrm{in}}+\psi_{\mathrm{sc}}\qquad r\to\infty \tag{20}$$

where

$$\psi_{\mathrm{in}}=\frac{\exp(-\mathrm{i}m_0\theta)}{\sqrt{k_1+k_2}}\begin{pmatrix}\mathrm{i}\sqrt{k_1}\\ \sqrt{k_2}\end{pmatrix}\varphi_{\mathrm{in}} \tag{21a}$$

with

$$\varphi_{\mathrm{in}}=\sum_{m=-\infty}^{+\infty}\mathrm{i}^m\sqrt{\frac{2}{\pi kr}}\cos\left(kr+\beta\ln 2kr-\frac{m\pi}{2}-\frac{\pi}{4}\right)\mathrm{e}^{\mathrm{i}m\theta} \tag{21b}$$

and

$$\psi_{\mathrm{sc}}=\sqrt{\frac{\mathrm{i}}{r}}\exp(\mathrm{i}kr+\mathrm{i}\beta\ln 2kr)f(\theta)\frac{\exp(-\mathrm{i}m_0\theta)}{\sqrt{k_1+k_2}}\begin{pmatrix}\mathrm{i}\sqrt{k_1}\\ \mathrm{e}^{\mathrm{i}\theta}\sqrt{k_2}\end{pmatrix} \tag{22}$$

with

$$f(\theta)=\sqrt{\frac{2}{\pi k}}\sum_j\exp(\mathrm{i}\eta_j)\sin\eta_j\mathrm{e}^{\mathrm{i}m\theta}=-\frac{\mathrm{i}}{\sqrt{2\pi k}}\sum_j(S_j-1)\mathrm{e}^{\mathrm{i}m\theta}. \tag{23}$$

In spite of the presence of the AB potential, the probability current density associated with a solution ψ is still given by

$$\boldsymbol{j} = c\psi^{\dagger}\boldsymbol{\alpha}\psi. \tag{24}$$

As in the case of a pure Coulomb field [1], one can show that the above ψ represents a correct scattering solution of which $\psi_{\rm in}$ is an incident wave and $\psi_{\rm sc}$ the scattered one, $f(\theta)$ is the scattering amplitude and the differential cross section is given by

$$\sigma(\theta) = |f(\theta)|^2. \tag{25}$$

The choice of a_j in equations (17) is thereby proved to be appropriate. Since $\sum_j e^{im\theta} = 2\pi\delta(\theta)$, we have for $\theta \neq 0$

$$f(\theta) = -\frac{\mathrm{i}}{\sqrt{2\pi k}}\sum_j S_j e^{\mathrm{i}m\theta}. \tag{26}$$

This result with S_j given by equation (19) is exact. However, it is even more difficult to sum up the above partial wave series than in the case of a pure Coulomb field.

Let us consider the case with $\gamma^2 \ll 1$ and try to work out a closed result. For electron–nucleus interaction we have $\gamma \approx \frac{Z}{137}$, so the above condition means that Z is small, say, $Z < 5$. In this case we may approximately replace s by $|j + \nu|$ in equation (19). Note that β also depends on γ and we do not make an approximation with it. Thus the result will possess some nonperturbative features in regard to the parameter γ. With the above approximation, S_j is replaced by

$$S_j^{\rm a} = e^{-\mathrm{i}\nu\pi}\frac{\Gamma(m+\nu+\frac{1}{2}-\mathrm{i}\beta)}{\Gamma(m+\nu+\frac{1}{2}+\mathrm{i}\beta)} - \mathrm{i}(\beta-\beta')e^{-\mathrm{i}\nu\pi}\frac{\Gamma(m+\nu+\frac{1}{2}-\mathrm{i}\beta)}{\Gamma(m+\nu+\frac{3}{2}+\mathrm{i}\beta)} \qquad (j>0) \tag{27a}$$

$$S_j^{\rm a} = e^{\mathrm{i}\nu\pi}\frac{\Gamma(|m|-\nu+\frac{1}{2}-\mathrm{i}\beta)}{\Gamma(|m|-\nu+\frac{1}{2}+\mathrm{i}\beta)} + \mathrm{i}(\beta-\beta')e^{\mathrm{i}\nu\pi}\frac{\Gamma(|m|-\nu-\frac{1}{2}-\mathrm{i}\beta)}{\Gamma(|m|-\nu+\frac{1}{2}+\mathrm{i}\beta)} \qquad (j<0). \tag{27b}$$

It can be shown that the first term in either equation is equal to $\exp(2\mathrm{i}\delta_m)$ where δ_m is the nonrelativistic phase shift of the mth partial wave, except for $m = 0$ when $-\frac{1}{2} < \nu < 0$. (The physical reason for the latter disagreement is not quite clear to us.) The second term is a relativistic correction which vanishes when $v_{\rm c}/c \to 0$. Substituting equations (27) into equation (26), we have an approximate result for $f(\theta)$:

$$f^{\rm a}(\theta,\nu) = f_0(\theta,\nu) + f_1(\theta,\nu) \tag{28}$$

where

$$\begin{aligned} f_0(\theta,\nu) = -\frac{\mathrm{i}}{\sqrt{2\pi k}}\Bigg[& e^{-\mathrm{i}\nu\pi}\frac{\Gamma(\frac{1}{2}+\nu-\mathrm{i}\beta)}{\Gamma(\frac{1}{2}+\nu+\mathrm{i}\beta)}F\left(1, \frac{1}{2}+\nu-\mathrm{i}\beta, \frac{1}{2}+\nu+\mathrm{i}\beta, e^{\mathrm{i}\theta}\right) \\ & + e^{\mathrm{i}\nu\pi}\frac{\Gamma(\frac{3}{2}-\nu-\mathrm{i}\beta)}{\Gamma(\frac{3}{2}-\nu+\mathrm{i}\beta)}e^{-\mathrm{i}\theta}F\left(1, \frac{3}{2}-\nu-\mathrm{i}\beta, \frac{3}{2}-\nu+\mathrm{i}\beta, e^{-\mathrm{i}\theta}\right)\Bigg] \end{aligned} \tag{29}$$

is the nonrelativistic partial wave result (somewhat different when $-\frac{1}{2} < \nu < 0$) and

$$\begin{aligned} f_1(\theta,\nu) = -\frac{\beta-\beta'}{\sqrt{2\pi k}}\Bigg[& e^{-\mathrm{i}\nu\pi}\frac{\Gamma(\frac{1}{2}+\nu-\mathrm{i}\beta)}{\Gamma(\frac{3}{2}+\nu+\mathrm{i}\beta)}F\left(1, \frac{1}{2}+\nu-\mathrm{i}\beta, \frac{3}{2}+\nu+\mathrm{i}\beta, e^{\mathrm{i}\theta}\right) \\ & - e^{\mathrm{i}\nu\pi}\frac{\Gamma(\frac{1}{2}-\nu-\mathrm{i}\beta)}{\Gamma(\frac{3}{2}-\nu+\mathrm{i}\beta)}e^{-\mathrm{i}\theta}F\left(1, \frac{1}{2}-\nu-\mathrm{i}\beta, \frac{3}{2}-\nu+\mathrm{i}\beta, e^{-\mathrm{i}\theta}\right)\Bigg] \end{aligned} \tag{30}$$

is the relativistic correction. In these equations $F(a_1, a_2, b_1, z)$ is the hypergeometric function [25]. Using their functional relations it can be shown that

$$f_0(\theta,\nu) = -\mathrm{i}\mathrm{e}^{-\mathrm{i}\nu\theta}\frac{\Gamma(\frac{1}{2}-\nu+\mathrm{i}\beta)\Gamma(\frac{1}{2}+\nu-\mathrm{i}\beta)}{\Gamma(\mathrm{i}\beta)\Gamma(\frac{1}{2}+\mathrm{i}\beta)}\frac{\exp(\mathrm{i}\beta\ln\sin^2\theta/2)}{\sqrt{2k}\sin\theta/2}$$
$$-\frac{\mathrm{i}}{\sqrt{2\pi k}}\left[\mathrm{e}^{\mathrm{i}\nu\pi}\frac{\Gamma(\frac{3}{2}-\nu-\mathrm{i}\beta)}{\Gamma(\frac{3}{2}-\nu+\mathrm{i}\beta)} - \mathrm{e}^{-\mathrm{i}\nu\pi}\frac{\Gamma(-\frac{1}{2}+\nu-\mathrm{i}\beta)}{\Gamma(-\frac{1}{2}+\nu+\mathrm{i}\beta)}\right]$$
$$\times\mathrm{e}^{-\mathrm{i}\theta}F(1,\tfrac{3}{2}-\nu-\mathrm{i}\beta,\tfrac{3}{2}-\nu+\mathrm{i}\beta,\mathrm{e}^{-\mathrm{i}\theta}) \qquad (31)$$

and

$$f_1(\theta,\nu) = -\left(1-\frac{\beta'}{\beta}\right)\frac{\Gamma(\frac{1}{2}+\nu-\mathrm{i}\beta)\Gamma(\frac{1}{2}-\nu+\mathrm{i}\beta)}{\Gamma(\mathrm{i}\beta)\Gamma(\frac{1}{2}+\mathrm{i}\beta)}\frac{\mathrm{e}^{-\mathrm{i}\theta/2-\mathrm{i}\nu\theta}\exp(\mathrm{i}\beta\ln\sin^2\theta/2)}{\sqrt{2k}}$$
$$+\frac{\beta-\beta'}{\sqrt{2\pi k}}\left[\mathrm{e}^{\mathrm{i}\nu\pi}\frac{\Gamma(\frac{1}{2}-\nu-\mathrm{i}\beta)}{\Gamma(\frac{3}{2}-\nu+\mathrm{i}\beta)} + \mathrm{e}^{-\mathrm{i}\nu\pi}\frac{\Gamma(-\frac{1}{2}+\nu-\mathrm{i}\beta)}{\Gamma(\frac{1}{2}+\nu+\mathrm{i}\beta)}\right]$$
$$\times\mathrm{e}^{-\mathrm{i}\theta}F(1,\tfrac{1}{2}-\nu-\mathrm{i}\beta,\tfrac{3}{2}-\nu+\mathrm{i}\beta,\mathrm{e}^{-\mathrm{i}\theta}) \qquad (32)$$

where $0 \leqslant \theta < 2\pi$. Note that these approximate expressions are well defined for the whole range of ν, and will be employed in the following discussions for the case when $|\nu|$ is close to $\frac{1}{2}$. Unlike the case with a pure Coulomb field, here we have additional terms in both $f_0(\theta,\nu)$ and $f_1(\theta,\nu)$, involving the hypergeometric functions. Thus closed forms are possible only for special ν when the additional terms vanish. This happens when $\nu = 0$ and $\nu = \frac{1}{2}$. Since the above discussions are not valid for $|\nu|$ close to $\frac{1}{2}$, we have now a closed result only when $\nu = 0$, or $q\phi/2\pi\hbar c$ takes on integer values. The result is the same as in the case of a pure Coulomb field:

$$f(\theta) = f^{\mathrm{a}}(\theta,0) = -\mathrm{i}\frac{\Gamma(\frac{1}{2}-\mathrm{i}\beta)}{\Gamma(\mathrm{i}\beta)}\frac{\exp(\mathrm{i}\beta\ln\sin^2\theta/2)}{\sqrt{2k}\sin\theta/2}\left[1-\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta/2}\sin\frac{\theta}{2}\left(1-\frac{\beta'}{\beta}\right)\right]. \qquad (33)$$

Since the result is more singular than $\delta(\theta)$ when $\theta \to 0$, the $\delta(\theta)$ term that is dropped above for $\theta \neq 0$ can indeed be dropped everywhere and the above expression is enough. The differential cross section reads

$$\sigma(\theta) = \frac{\beta\tanh\beta\pi}{2k\sin^2\theta/2}\left(1-\frac{v_{\mathrm{c}}^2}{c^2}\sin^2\frac{\theta}{2}\right) = \frac{\kappa\tanh(\pi\kappa/\hbar v_{\mathrm{c}})}{2\mu v_{\mathrm{c}}^2\sin^2\theta/2}\left(1-\frac{v_{\mathrm{c}}^2}{c^2}\sin^2\frac{\theta}{2}\right)\left(1-\frac{v_{\mathrm{c}}^2}{c^2}\right)^{\frac{1}{2}} \qquad (34)$$

where the first factor in the last expression is the exact nonrelativistic result†, and the subsequent ones are due to the relativistic effect. We make some remarks similar to those made in [1]. First, we have not made any approximation in regard to the incident velocity, so the result is valid for high-energy collision. It is obvious that the relativistic correction becomes significant when v_{c} is comparable with c. Second, though the above result holds for small γ only, it involves a nonperturbative factor $\tanh\beta\pi$ (note that $\beta \propto \gamma$). Thus the result cannot be obtained from perturbative QED at the tree level. Moreover, if m_0 or ϕ is large, perturbative QED seems not to be applicable, but the above calculations hold as well.

If ν is close to $\frac{1}{2}$ $(-\frac{1}{2})$ but $|\gamma| < |\frac{1}{2} \pm \nu|$, then the above discussions are still valid, but then the replacement of $S_{-1/2}$ $(S_{1/2})$ by $S^{\mathrm{a}}_{-1/2}$ $(S^{\mathrm{a}}_{1/2})$ is a poor approximation. In this case some corrections are necessary. Since no closed result is available, we do not discuss it in detail.

† In the nonrelativistic case when $\nu = 0$ and $m_0 \neq 0$, we obtained an interference term in the cross section in additional to the result for a pure Coulomb field, because we excluded the s-wave solution, which is slightly singular at the origin [3]. Now it seems that the s-wave is acceptable and that the interference term is not necessary, because the potentials themselves are rather singular at the origin. Indeed, in the relativistic case, the solutions with $j = \pm\frac{1}{2}$ are much more singular, and the singularity is essentially the same as that in a pure Coulomb field.

Now we turn to the case when ν is very close to $\frac{1}{2}$ such that $|\gamma| > |\frac{1}{2} - \nu|$ (the typical case is $\nu = \frac{1}{2}$); then the above discussions have to be modified. We will discuss this in some detail. (The other case with ν very close to $-\frac{1}{2}$ can be discussed in a similar way and will be omitted.) The crucial point is that s becomes imaginary when $j = -\frac{1}{2}$. (A larger γ causes the same difficulty for other values of j.) So the solutions of equation (13) are now given by

$$\begin{aligned} u^{(1)}_{-1/2}(\rho) &= a^{(1)}_{-1/2} \exp(\mathrm{i}\gamma' \ln \rho)\Phi(-\mathrm{i}\beta + \mathrm{i}\gamma', 1 + 2\mathrm{i}\gamma', -2\mathrm{i}\rho) \\ u^{(2)}_{-1/2}(\rho) &= a^{(2)}_{-1/2} \exp(-\mathrm{i}\gamma' \ln \rho)\Phi(-\mathrm{i}\beta - \mathrm{i}\gamma', 1 - 2\mathrm{i}\gamma', -2\mathrm{i}\rho) \end{aligned} \tag{35}$$

where

$$\gamma' = \sqrt{\gamma^2 - (\tfrac{1}{2} - \nu)^2}. \tag{36}$$

Here the two solutions have essentially the same singularity at the origin. They oscillate very rapidly and tend to no limit when $\rho \to 0$. In the conventional opinion of quantum mechanics, such solutions are not acceptable [26, 27]. One possible way of resolving the problem is to cut off the singular potentials (which represent some idealization) at some small radius, but this causes some mathematical difficulties. On the other hand, some authors have attempted to handle such solutions in some convenient way, and for bound states proposals have been put forward to choose the appropriate linear combination of the two solutions [28, 29]. For scattering solutions the situation seems more involved. Here we will resolve the problem by some simple consideration. The solutions $v^{(1)}_{-1/2}(\rho)$ and $v^{(2)}_{-1/2}(\rho)$ can be easily obtained from equation ($11a$) and the above u. We will not write them down. We take

$$a^{(1)}_{-1/2} = A\left[\beta' + \mathrm{i}\left(\frac{1}{2} - \nu\right)\right] \frac{\Gamma(-\mathrm{i}\beta + \mathrm{i}\gamma')}{\Gamma(1 + 2\mathrm{i}\gamma')} \exp\left(\frac{\beta\pi}{2} + \frac{\gamma'\pi}{2} + \mathrm{i}\gamma' \ln 2 - \mathrm{i}\frac{\pi}{4}\right) \tag{37}$$

then the asymptotic forms for $f^{(1)}_{-1/2}(r)$ and $g^{(1)}_{-1/2}(r)$ are given by equations (18) with $S_{-1/2}$ replaced by

$$S^{(1)} = \left[\beta' + \mathrm{i}\left(\frac{1}{2} - \nu\right)\right] \frac{\Gamma(-\mathrm{i}\beta + \mathrm{i}\gamma')}{\Gamma(1 + \mathrm{i}\beta + \mathrm{i}\gamma')} \exp(\gamma'\pi). \tag{38}$$

However, $|S^{(1)}| \neq 1$, so $S^{(1)}$ cannot be expressed as a phase factor. We then take $a^{(2)}_{-1/2}$ by replacing γ' by $-\gamma'$ in $a^{(1)}_{-1/2}$, then the asymptotic forms for $f^{(2)}_{-1/2}(r)$ and $g^{(2)}_{-1/2}(r)$ are given by equations (18) with $S_{-1/2}$ replaced by $S^{(2)}$, which is obtained by replacing γ' by $-\gamma'$ in $S^{(1)}$. As the two solutions above are equally preferable, we take the mean as the required solution. Then the asymptotic forms for $f_{-1/2}(r)$ and $g_{-1/2}(r)$ are given by equations (18) with $S_{-1/2}$ replaced by

$$S = \tfrac{1}{2}[S^{(1)} + S^{(2)}]. \tag{39}$$

The solutions with $j \neq -\frac{1}{2}$ are all the same as those given before. So we finally obtain the scattering amplitude (for $\theta \neq 0$)

$$f(\theta) = -\frac{\mathrm{i}}{\sqrt{2\pi k}}\left[\sum_{j \neq -1/2} S_j \mathrm{e}^{\mathrm{i}m\theta} + S\mathrm{e}^{-\mathrm{i}\theta}\right]. \tag{40}$$

This is a very complicated result. When $\gamma^2 \ll 1$, one can make approximations as before, but closed results are available only when $\nu = \frac{1}{2}$, or $q\phi/2\pi\hbar c$ takes on half-integer values. We have then

$$f(\theta) = f^{\mathrm{a}}\left(\theta, \frac{1}{2}\right) - \frac{\mathrm{i}}{\sqrt{2\pi k}}(S - S^{\mathrm{a}}_{-1/2})\mathrm{e}^{-\mathrm{i}\theta}. \tag{41}$$

Note that $S^{\mathrm{a}}_{-1/2}$ is well defined in equation (27*b*). Since we have neglected γ^2 in calculating $f^{\mathrm{a}}(\theta, \frac{1}{2})$, we can also neglect it in calculating S. It then turns out that $S = S^{\mathrm{a}}_{-1/2}$ to the first order in γ. So we have

$$f(\theta) = f^{\mathrm{a}}\left(\theta, \frac{1}{2}\right) = -\mathrm{e}^{-\mathrm{i}\theta/2}\frac{\beta\Gamma(-\mathrm{i}\beta)}{\Gamma(\frac{1}{2}+\mathrm{i}\beta)}\frac{\exp(\mathrm{i}\beta\ln\sin^2\theta/2)}{\sqrt{2k}\sin\theta/2}\left[1-\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta/2}\sin\frac{\theta}{2}\left(1-\frac{\beta'}{\beta}\right)\right]. \tag{42}$$

The differential cross section reads

$$\sigma(\theta) = \frac{\beta\coth\beta\pi}{2k\sin^2\theta/2}\left(1-\frac{v_{\mathrm{c}}^2}{c^2}\sin^2\frac{\theta}{2}\right) = \frac{\kappa\coth(\pi\kappa/\hbar v_{\mathrm{c}})}{2\mu v_{\mathrm{c}}^2\sin^2\theta/2}\left(1-\frac{v_{\mathrm{c}}^2}{c^2}\sin^2\frac{\theta}{2}\right)\left(1-\frac{v_{\mathrm{c}}^2}{c^2}\right)^{\frac{1}{2}} \tag{43}$$

where the first factor in the last expression is the exact nonrelativistic result [3], and the subsequent ones are due to the relativistic effect. The remarks made under equation (34) are also applicable here.

Finally we give some results concerning the bound states. If $|\gamma| < |\frac{1}{2} \pm \nu|$, the solutions can be obtained without difficulty. The energy levels are

$$E_{nj} = \frac{\mu c^2}{[1+\gamma^2/(n+s)^2]^{1/2}} \tag{44}$$

where $n = 0, 1, 2, \ldots$ is a radial quantum number, and s is defined in equations (14), and depends on j. The level with $n = 0$ is not degenerate regardless of ν, since one can show that only solutions with $j > 0$ are possible in this case. When $n > 0$, the degeneracy depends on ν. If $\nu \neq 0$, there is no degeneracy. If $\nu = 0$, the energy level depends on $|j|$ rather than j, and solutions with both positive and negative j exist. So the level is double degenerate. It is remarkable that there are no negative energy levels. The wavefunctions are given in terms of confluent hypergeometric functions. Since the results are complicated we will not write them down. If $|\gamma| > |\frac{1}{2} \pm \nu|$, s will become imaginary for some j. For such values of j, special treatment [28, 29] of the solution is necessary, and the results are rather involved. We will not go into further details here.

In conclusion, we have calculated the scattering amplitude and differential cross section for fast particles with ABC interaction in two dimensions. Exact results are given in partial wave series in general cases. Approximate results in closed forms are given in special cases. Though approximate, the results exhibit some nonperturbative features and cannot be obtained from the lowest-order contribution of perturbative QED. We have also discussed the bound-state solutions and given some results.

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